An Averaging Method for Nash Games with Shared Decision Variables*

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Abstract

This short communication concerns a class of Nash games in which the participants share some of the decision variables. An averaging method is developed in order to transform these games into standard Nash games. This technique is successfully applied to solve an example multileader, single-follower game described by Pang and Fukushima.

1 Introduction

A Nash game [7, 8] is a noncooperative game generally played by n individuals who each select a strategy to maximize their own profit given the strategies chosen by the other participants. A Nash equilibrium is a set of strategies where each participant has chosen a strategy that maximizes the player's individual profit. That is, no player can make a unilateral change in strategy that results in larger profit. For a two-player Nash game, $(x^{1,*}, x^{2,*})$ is a Nash equilibrium if and only if

$$\begin{array}{lll} x^{1,*} & \in & \arg\max_{x^1 \in X_1} f_1(x^1, x^{2,*}) \\ x^{2,*} & \in & \arg\max_{x^2 \in X_2} f_2(x^{1,*}, x^2), \end{array}$$

where f_1 and f_2 are the profit functions for the first and second player and X_1 and X_2 are *polyhedral* sets restricting the strategies that can be chosen. If $f_1(\cdot, x^2)$ is a concave function for every $x^2 \in X_2$ and if $f_2(x^1, \cdot)$ is a concave function for every $x^1 \in X_1$, then a standard technique is to write down the first-order optimality conditions for each optimization problem and put them all together to produce the variational inequality

$$\begin{array}{lll} X_1 \ni x^1 & \perp & -\nabla_{x^1} f_1(x^1, x^2) \\ X_2 \ni x^2 & \perp & -\nabla_{x^2} f_2(x^1, x^2), \end{array}$$

where the notation from [2] has been used to describe the polyhedrally-constrained variational inequality. Any solution to this variational inequality is a Nash equilibrium for the

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original game and vice versa because of the concavity assumption made for the objective functions and the polyhedrality of the constraint set. If the concavity assumption is not satisfied, an attempt can still be made to solve the variational inequality, but the solution is only a Nash critical point.

This short communication concerns a modified version of the standard Nash game in which an additional set of shared decision variables are incorporated into the optimization problems. Such games are encountered when solving multileader, single-follower games [10] in which each leader solves an optimization problem where the constraints model the response of the single follower. If the leaders maximize their objective function subject to how they think the follower will respond, then the problem is a standard Nash game because the perception of the response can be different for each leader. If the follower must make a single shared decision, however, the problem is no longer a standard Nash game because the follower's decision is then shared by all of the leaders.

In particular, $(x^{1,*}, x^{2,*}, y^*)$ is a Nash equilibrium for a two-player Nash game with shared decision variables if and only if

$$\begin{array}{lcl} (x^{1,*}, y^*) & \in & \arg\max_{x^1 \in X_1, y \in Y} f_1(x^1, x^{2,*}, y) \\ (x^{2,*}, y^*) & \in & \arg\max_{x^2 \in X_2, y \in Y} f_2(x^{1,*}, x^2, y), \end{array} \tag{1}$$

where Y is a *polyhedral* set restricting the possible values for the shared decision variables. As with standard Nash games, if $f_1(\cdot, x^2, \cdot)$ is a concave function for every $x^2 \in X_2$ and if $f_2(x^1, \cdot, \cdot)$ is a concave function for every $x^1 \in X_1$, then the first-order optimality conditions for each optimization problem can be written to produce the *nonsquare* variational inequality

$$\begin{array}{lcl} X_1 \ni x^1 & \perp & -\nabla_{x^1} f_1(x^1, x^2, y) \\ X_2 \ni x^2 & \perp & -\nabla_{x^2} f_2(x^1, x^2, y) \\ Y \ni y & \perp & -\nabla_y f_1(x^1, x^2, y) \\ Y \ni y & \perp & -\nabla_y f_2(x^1, x^2, y). \end{array}$$

This variational inequality is nonsquare because the shared decision variables appear into two different variational relationships. An alternative formulation allows each player to make independent decisions y^1 and y^2 and adds a constraint to the problem that $y^1 = y^2$. Such a modification still results in a nonsquare variational inequality because this problem has more constraints than decision variables. These nonsquare systems are difficult to solve for algorithms, such as PATH [1, 3], that were designed for square systems.

Section 2 discusses an averaging method for Nash games with shared decision variables that leads to a square variational inequality when the first-order optimality conditions are constructed. Section 3 then demonstrates the effectiveness of the averaging technique on the multileader, single-follower game described in [10].

2 Averaging Method

A square variational inequality can be produced for the Nash game with shared decision variables by letting all the leaders make their own decision for the shared variables and averaging these decisions in the objective function. The averaged Nash game is then to find

$(x^{1,*}, x^{2,*}, y^{1,*}, y^{2,*})$ such that

$$\begin{array}{rcl} (x^{1,*}, y^{1,*}) & \in & \arg\max_{x^1 \in X_1, y^1 \in Y} f_1\left(x^1, x^{2,*}, \frac{y^1+y^{2,*}}{2}\right) \\ (x^{2,*}, y^{2,*}) & \in & \arg\max_{x^2 \in X_2, y^2 \in Y} f_2\left(x^{1,*}, x^2, \frac{y^{1,*}+y^2}{2}\right). \end{array} \tag{2}$$

If the objective function for each player is a concave function for every choice of the fixed variables, then solutions to the averaged game can be characterized by the *square* variational inequality constructed from the first-order optimality conditions for each optimization problem

$$\begin{array}{rcl} X_1 \ni x^1 & \perp & -\nabla_{x^1} f_1\left(x^1, x^2, \frac{y^1+y^2}{2}\right) \\ X_2 \ni x^2 & \perp & -\nabla_{x^2} f_2\left(x^1, x^2, \frac{y^1+y^2}{2}\right) \\ Y \ni y^1 & \perp & -\nabla_{y^1} f_1\left(x^1, x^2, \frac{y^1+y^2}{2}\right) \\ Y \ni y^2 & \perp & -\nabla_{y^2} f_2\left(x^1, x^2, \frac{y^1+y^2}{2}\right). \end{array}$$

Any convex combination of the shared variables can be used in this averaging procedure.

Theorem 2.1 *If Y is a convex set, then the following hold:*

1. *If $(x^{1,*}, x^{2,*}, y^*)$ solves (1), then $(x^{1,*}, x^{2,*}, y^*, y^*)$ solves (2).*
2. *If $(x^{1,*}, x^{2,*}, y^{1,*}, y^{2,*})$ solves (2), then $(x^{1,*}, x^{2,*}, \frac{y^{1,*}+y^{2,*}}{2})$ solves (1).*

Proof: Let $(x^{1,*}, x^{2,*}, y^*)$ solve (1) and assume that $(x^{1,*}, x^{2,*}, y^*, y^*)$ does not solve (2). The latter statement means that one of the players can increase their objective function value by switching to a different strategy $(\bar{x}^1, x^{2,*}, \bar{y}, y^*)$. However, $(\bar{x}^1, x^{2,*}, \frac{\bar{y}+y^*}{2})$ is feasible for (1) because Y is a convex set and contradicts the fact that $(x^{1,*}, x^{2,*}, y^*)$ solves (1) because the objective function value for the first player at the new point increases. Therefore, the assumption was false and $(x^{1,*}, x^{2,*}, y^*, y^*)$ solves (2).

What remains to be shown is that if $(x^{1,*}, x^{2,*}, y^{1,*}, y^{2,*})$ is a solution to (2) then $(x^{1,*}, x^{2,*}, \frac{y^{1,*}+y^{2,*}}{2})$ is a solution to (1). However, if $(x^{1,*}, x^{2,*}, y^{1,*}, y^{2,*})$ is a solution to (2), then $(x^{1,*}, x^{2,*}, \frac{y^{1,*}+y^{2,*}}{2}, \frac{y^{1,*}+y^{2,*}}{2})$ is also a solution to (2) since the new point is feasible because Y is a convex set and the objective function values are the same. Clearly, $(x^{1,*}, x^{2,*}, \frac{y^{1,*}+y^{2,*}}{2})$ then solves (1). ■

Therefore, the averaged game and the game with shared decision variables are equivalent under a suitable mapping. The costs paid are that many new variables can be introduced into the averaged game and the averaged game can have an infinite number of solutions obtained by selecting different partitions for the shared decision variables.

Note that while the solution computed for the original game is correct, any Lagrange multipliers added when the Karush-Kuhn-Tucker conditions for the variational inequality are applied to produce an equivalent nonlinear complementarity problem may need to be recalculated for the averaged solution. The correct Lagrange multipliers can be found, for example, by calculating an active set and solving an auxiliary optimization problem.

This result can be extended to generalized Nash games with shared decision variables. Each optimization problem in a generalized Nash game can include constraints that depend on the strategies selected by all of the players [9]. Formally, the generalized Nash game with shared decision variables is to find $(x^{1,*}, x^{2,*}, y^*)$ such that

$$\begin{array}{rcll} (x^{1,*}, y^*) & \in & \arg\max_{x^1 \in X_1, y \in Y} & f_1(x^1, x^{2,*}, y) \\ & & \text{subject to} & g_1(x^1, x^{2,*}, y) \geq 0 \\ (x^{2,*}, y^*) & \in & \arg\max_{x^2 \in X_2, y \in Y} & f_2(x^{1,*}, x^2, y) \\ & & \text{subject to} & g_2(x^{1,*}, x^2, y) \geq 0, \end{array}$$

where X_1, X_2, and Y are polyhedral sets and f_1, f_2, g_1, and g_2 are concave functions. The averaged problem is to find $(x^{1,*}, x^{2,*}, y^{1,*}, y^{2,*})$ such that

$$\begin{array}{rcll} (x^{1,*}, y^{1,*}) & \in & \arg\max_{x^1 \in X_1, y^1 \in Y} & f_1(x^1, x^{2,*}, \frac{y^1 + y^{2,*}}{2}) \\ & & \text{subject to} & g_1(x^1, x^{2,*}, \frac{y^1 + y^{2,*}}{2}) \geq 0 \\ (x^{2,*}, y^{2,*}) & \in & \arg\max_{x^2 \in X_2, y^2 \in Y} & f_2(x^{1,*}, x^2, \frac{y^{1,*} + y^2}{2}) \\ & & \text{subject to} & g_2(x^{1,*}, x^2, \frac{y^{1,*} + y^2}{2}) \geq 0 \end{array}$$

The same averaging argument used in the proof of Theorem 2.1 is valid for these generalized Nash games with shared decision variables because if $y^{1,*}$ and $y^{2,*}$ are solutions to the averaged game, then $\bar{y^1} = \bar{y^2} = \frac{y^{1,*} + y^{2,*}}{2}$ is also a solution to the averaged game, since the new point is feasible and has the same objective function value.

Furthermore, the averaging method can be extended to games with more than two players by averaging the shared decisions made by all of the players in the game.

3 Numerical Test

The multileader, single-follower game posed in [10] was used to test the effect of the averaging method when computing an equilibrium to a game with shared decision variables. This game models an electric power market with endogenous arbitragers. The problem consists of four entities: the power producers that generate power to maximize their profits, the independent service operator that moves the generated power around the electricity network to maximize revenue, an arbitrager that buys and sells power in an attempt to make a profit from the sales, and the transmission market that sets the cost of transmitting power along the arcs in the electricity network by using a market clearing condition. The arbitrager is the single follower in the game and responds to the decisions made by the producers. The arbitrager is allowed to make only a single shared decision because this decision is used in the market clearing condition.

The exact descriptions of the optimization problems for each player can be found in Section 5.2 of [10] and are not repeated here. The optimization problem solved by each power producer is nonconvex because of the presence of a complementarity condition in the constraints, while the all of the other agents solve linear programs. The averaging method still works for this game by restricting the arbitrager's decision to the nonnegative orthant and using the averaged value in the complementarity conditions. The complementarity condition in the constraints for each of the producers can cause problems because of constraint qualification violations [11]. Therefore, the producer problems are reformulated by using the technique in [6] for sequential quadratic programming methods, and the first-order optimality conditions are constructed for the reformulated problem. Because of the nonconvexity of the constraints, a solution to the variational inequality may be only a Nash critical point for the original game.

Two models for this electric power market model were implemented in the AMPL modeling language [5] to test the effect of the averaging method when computing a Nash critical point for a game with shared decision variables. The first model implements the original nonsquare formulation; the second model implements the square formulation obtained by using the averaging method. Both models are available for download from `http://www.mcs.anl.gov/~tmunson/models`. The PATH algorithm [1, 3] was used in an attempt to solve the resulting complementarity problems. The default settings were used for this algorithm in all of the tests. The AMPL presolve was also turned off because it can sometimes produce a nonsquare complementarity problem from a square system. A starting point consisting of the zero vector for all quantities and multipliers was used for testing.

As expected, the nonsquare complementarity problem for the original formulation is very difficult to solve. PATH gives up on solving this problem and admits failure after 187 major iterations and three algorithm restarts [4]. The best point found had a residual of 1.5, which is far from a solution to the problem. The multipliers on the complementarity constraints did not appear to diverge; the maximum element in the best point found was under 5,000. Rather, the reason for failure appears to be that the problem is not square.

For the averaged model, however, PATH converged to a very accurate solution in 8 major iterations. The residual at the solution reported was 1.5×10^{-12} and the multipliers on the complementarity constraints were all less than 100. The Newton direction was accepted at every iteration with a step size of one, and fast convergence was observed. At the solution reported, $y^{1,*}$ and $y^{2,*}$ were different, but the averaged value is a solution to the original game.

4 Conclusion

The averaging method is a useful technique for solving Nash games with shared decision variables because it produces a square variational inequality when the first-order optimality conditions are taken for each of the optimization problems solved by the participants. While the models generated by the averaging method are more computationally tractable, the cost paid is that many new variables are potentially introduced, therefore impacting the sparsity of the Jacobian and solution time. Furthermore, some algorithms may have difficulty solving the averaged game because it can have an infinite number of solutions. The PATH algorithm, however, seems to have no difficulty solving the averaged game.

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